International Journal of Theoretical Physics, Vol. 17, No. 5 (1978), pp. 369–376

The Triviality of Continuous Multipliers for the Real Line

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Received September 30, 1977

It is proved without resort to "calculus methods" that every continuous group multiplier for **R** can be reduced to the identity by a continuous "remultiplication." The method introduced may generalize to infinite-dimensional Abelian groups such as occur in analyzing the projective representations of the Bondi–Metzner–Sachs (BMS) group.

1. INTRODUCTION

Projective, as opposed to true, representations of Lie groups made an unexpected entrance into physics with the Dirac spinors. According to later analysis the appearance of such representations, characterized by the relation

$$R(g)R(h) = \sigma(g, h)R(gh) \tag{1.1}$$

is rooted in the fact that a quantum-mechanical-state vector is determined only up to an overall phase. Accordingly, a "symmetry group" acts in the Hilbert space of the system in terms of unitary (or "antiunitary" in certain cases) operators related by (1.1) (in which σ is now a complex number of absolute value 1).

Just as the Galilean group is the symmetry group of classical physics, the so-called BMS (Bondi–Metzner–Sachs) group is, from a certain point of view, that of general relativity. Therefore the classification of its irreducible unitary projective representations is of great interest. In particular, there is the mathematical question as to what extent the class of projective representations is broader than that of true representations [characterized by (1.1) with $\sigma \equiv 1$].

Recent work of McCarthy[1] (1977) has almost answered this question, but a technical problem remains. The present paper stems from his suggestion (private communication) that a solution by noncalculus methods of the analogous problem for the group **R** might provide a clue to the BMS case. In fact, the result on which the present method is based (the theorem of Section 2) probably will generalize to infinite dimensions, although I have not been able to prove this.

There seem to be two distinct ways in which nontrivial group multipliers, as σ is known, can arise, at least for a continuous group G. The first has to do with the overall topology of the group, the second only with its local structure. The first, as exemplified by the relation between the double connectivity of the three-dimensional rotation group and the existence of half-integral angular momentum, is probably the most widely recognized. But the second is also physically important, for instance, in representations of the Galilean group, or equivalently in the Weyl form of the Heisenberg commutation relations (Jauch, 1968; Weyl, 1950). For the BMS group it is the analysis of the second way which is causing trouble. Let us be more concrete.

For physical reasons one wants *continuous* projective representations of G, that is, representations which are continuous maps of G into the space of projective transformations of the representation space H. But a projective transformation is equivalent modulo phase to a unitary operator, and Wigner (see Bargmann, 1954) has shown that it is always possible, at least locally, to choose the phases in such a way that both σ and R in equation (1.1) are continuous functions.

By further changing the phases of R one can hope to reduce R to a true representation. If

$$\hat{R}(g) = R(g)/\lambda(g) \tag{1.2}$$

then σ in (1.1) becomes $\hat{\sigma}$, where

$$\hat{\sigma}(g, h) = \frac{\lambda(g, h)}{\lambda(g)\lambda(h)}\,\sigma(g, h) \tag{1.3}$$

is said to be equivalent to σ. The condition that R be equivalent to a true representation is thus that $\sigma(g, h)$ be of the form

$$\sigma(g, h) = \lambda(g)\lambda(h)/\lambda(gh) \tag{1.4}$$

so that $\hat{\sigma} \equiv 1$.

Since it would not be of much use to reduce the problem of finding the continuous projective representations of G to that of finding its discontinuous true representations, we want λ to be continuous. The problem we will solve

[1] See McCarthy also for background and other references on group multipliers.

is as follows: Given a continuous multiplier σ for $G = \mathbf{R}$, to find a continuous "remultiplier" which trivializes σ [equation (1.4)]. In the BMS case in question it has been proved (McCarthy) that (1.4) can be solved, but not necessarily by a *continuous* λ. The usual methods which prove continuity for $G = \mathbf{R}$ fail for infinite-dimensional groups since they rely ultimately on some form of local compactness [see McCarthy].

2. THE RESULT FOR $G = \mathbf{R}$

Let σ: $\mathbf{R} \times \mathbf{R} \to \mathbf{C}$ be any group multiplier for $G = \mathbf{R}(+)$, the additive group of the real line. By definition σ is any (nowhere zero) map fulfilling the identity $\forall x, y, z \in \mathbf{R}$:

$$\sigma(x, y)\sigma(x + y, z) = \sigma(x, y + z)\sigma(y, z) \tag{2.1}$$

which is the translation via (1.1) of the associativity of group multiplication [which is just addition when $G = \mathbf{R}(+)$]. Conversely, any solution of (2.1) can be associated to a projective representation R of $\mathbf{R}$ whose substratum is a space H of complex functions on $\mathbf{R}$ and whose action can be indicated by

$$R(x)|y\rangle = \sigma(x, y)|x + y\rangle \tag{2.2}$$

where $|x\rangle$ represents a function concentrated at x. In terms of R our task is to determine λ so that $\hat{R}$ in (1.2) becomes a true representation. For convenience we will normalize σ and R so that, equivalently,

$$R(0) = 1 \tag{2.3}$$

$$\sigma(0, x) = \sigma(x, 0) = 1 \tag{2.4}$$

As a first step in finding λ consider any rational number q and the cyclic subgroup $(q) := q\mathbf{Z}$ of Q generated by q. If R is to be a true representation, then

$$\hat{R}(mq) = \hat{R}(q)^m \qquad m \in \mathbf{Z} \tag{2.5}$$

which means that $\lambda(q)$ determines $\lambda(s)$ for all s in (q). In particular, if $q = (n!)^{-1}$, then a choice of $\hat{R}(1/n!)$ defines by (2.5) a self-consistent choice of $\hat{R}(s)$ and thereby of $\lambda(s)$ for all s in $(1/n!)$. We can therefore choose inductively

$$\hat{R}(1), \hat{R}(1/2), \hat{R}(1/6), \ldots$$

to produce a self-consistent set of $\hat{R}$'s. At each stage the requirement

$$\hat{R}(1/n!)^n = \hat{R}(1/(n-1)!)$$

determines $\lambda(1/n!)$ up to multiplication by an nth root of unity. This process produces a self-consistent choice of λ for all s in

$$\bigcup_{n=0}^{\infty} \left(\frac{1}{n!}\right) = \mathbf{Q}$$

(the Appendix contains an explicit expression for λ). Since Q is dense in $\mathbf{R}$ this will be enough to define λ everywhere—if λ can be chosen with the necessary continuity.

In trying to prove the continuity (on $\mathbf{Q}$) of λ as we have constructed it so far, we would be troubled by the arbitrary choices involved in that construction. By working instead with $\mu = \log \lambda$ we can escape this difficulty since the solution of the log of (1.4),

$$\rho(x, y) = \mu(x) + \mu(y) - \mu(x + y) \tag{2.6}$$

where

$$\sigma(x, y) = e^{\rho(x,y)} \tag{2.7}$$

$$\lambda(x) = e^{\mu(x)} \tag{2.8}$$

involves on $\mathbf{Q}$ only one free choice.

However, this brings in the nuisance that a solution of (1.4) (which we have established) does not immediately give one of (2.6)–(2.8) since the passage to logarithms is not unique. Nevertheless, I claim that (2.6) *is* soluble on $\mathbf{Q}$.

In the first place, because σ is continuous (2.7) determines a unique continuous ρ with $\rho(0, 0) = 0$. We thus get a continuous map $\rho: \mathbf{R} \times \mathbf{R} \to \mathbf{C}$ obeying [from (2.1)]

$$\rho(x, y) + \rho(x + y, z) = \rho(x, y + z) + \rho(y, z) \tag{2.9}$$

Notice now that the above construction on $\mathbf{Q}$ of λ from σ rests ultimately only on (2.1) and the fact that nth roots exist in the multiplicative group $\mathbf{C} - \{0\}$. But (2.9) is the same as (2.1) only with the additive group $\mathbf{C}$ replacing $\mathbf{C} - \{0\}$. Since nth "roots" (i.e., division by n) exist also in $\mathbf{C}(+)$, the corresponding conclusion can be drawn, namely, the existence of $\mu: \mathbf{Q} \to \mathbf{C}$ satisfying (2.6).

We want to extend μ to a continuous remultiplier on $\mathbf{R}$. Call a function $f: \mathbf{Q} \to \mathbf{C}$ *completably continuous* iff it extends to a continuous function $\bar{f}: \mathbf{R} \to \mathbf{C}$. We will show that μ, which we consider henceforth only as a function with domain $\mathbf{Q}$, is completably continuous.

The notion key to this demonstration is the following. A function f from a subset S of $\mathbf{R}$ into $\mathbf{C}$ will be said to be PA (preadditive) if it satisfies: $\forall x, y, u, v \in S$:

$$x + y = u + v \Rightarrow f(x) + f(y) = f(u) + f(v) \qquad \text{(PA)}$$

If equality is weakened to "difference $\leqslant \epsilon$" f will be called ϵ-PA on S.

Lemma 1. $\forall \xi \in \mathbf{R}, \forall \epsilon > 0, \exists \delta > 0$ such that μ is ϵ-PA on

$$I := (\xi - \delta, \xi + \delta) \subset \mathbf{Q}$$

Proof. Because of our normalization of σ, $\rho(0, x) = \rho(x, 0) = 0$. From (2.9) $\rho(x + y, z) - \rho(x, y + z) = \rho(y, z) - \rho(x, y)$. Substituting (2.6),

$$[\mu(x + y) + \mu(z)] - [\mu(z + y) + \mu(x)] = \rho(y, z) - \rho(x, y) \quad (2.10)$$

Choose δ so that $|\rho(s, t)| < \epsilon/2$ for s in I and $|t| < 2\delta$. If $x, y, u, v \in I$ and $x + y = u + v$, then the substitution $z \to y$, $x \to u$, $y \to x - u$ yields

$$[\mu(x) + \mu(y)] - [\mu(v) + \mu(u)] = \rho(x - u, y) - \rho(u, x - u)$$

which is less than ϵ by our choice of δ. □

Lemma 2. If μ is not completable at $\xi \in \mathbf{R}$, then it is unbounded (on $\mathbf{Q}$) in every neighborhood of ξ.

Proof. Let $N \subset \mathbf{Q}$ be any neighborhood containing ξ and M any positive integer. Since μ is not completable at ξ there is $\epsilon > 0$ such that

$$(\forall\delta > 0)(\exists x, u)\colon |x - \xi|, |u - \xi| < \delta \qquad \text{and} \qquad \mu(u) - \mu(x) > \epsilon \quad (2.11)$$

Pick δ so that μ is $(\epsilon/2)$-PA on $I \coloneqq (\xi - 2M\delta, \xi + 2M\delta)$ and so that $I \subset N$. If u and x are as in (2.11), then $(\epsilon/2)$-PA with $u - x = y - v \eqqcolon z$, together with (2.11), yields $\mu(v + z) - \mu(v) > \epsilon - \epsilon/2 = \epsilon/2$. By induction, therefore (clearly, $x + Mz \in \mathbf{Q}$ since $x, u \in \mathbf{Q}$),

$$\mu(x + Mz) - \mu(x) > M\epsilon/2 > \max(|\mu(x + Mz)|, |\mu(x)|) > M\epsilon/4$$

Since both x and $x + Mz \in I \subset N$ while M was arbitrary, we are done. □

Theorem. Let S be any finite grid of evenly spaced points of $\mathbf{R}$. If the real function f is ϵ-PA on S, then f is approximated on S to within $\epsilon/2$ by some straight line $f_0(x) = Ax + B$.

Proof. Let S comprise $x_0 < x_1 < x_2 < \cdots < x_N$. We prove by induction on N the slightly strengthened assertion that f is within $[(N - 1)/N]\epsilon$ of the straight line through the points $(x_0, f(x_0))$ and $(x_N, f(x_N))$. The starting case $N = 1$ is trivial. In carrying out the inductive step it is convenient to work with $S = \{0, 1, \ldots, N\}$, fix k such that $0 < k < N$, and examine the value $w \coloneqq f(k)$.

Let $N = mk + s$ with $0 < s \leqslant k$ (*not* $0 \leqslant s < k$) and consider the two grids

$$\{0, k, 2k, \ldots, mk\} \qquad \text{and} \qquad \{N - k, N - k + 1, \ldots, N - 1, N\}$$

both of which have fewer than $N + 1$ points. The inductive hypothesis for the first grid supplies

$$w \leqslant \frac{m-1}{m} f(0) + \frac{1}{m} f(mk) + \frac{m-1}{m}\epsilon$$

The inductive hypothesis for the second grid supplies

$$f(mk) \leqslant \frac{s}{k} f(N-k) + \frac{k-s}{k} f(N) + \frac{k-1}{k}\epsilon$$

Finally, ϵ-PA for the points $0, k, N-k, N$ says

$$w + f(N-k) \leqslant \epsilon + f(0) + f(N)$$

Strung together, these three inequalities become, after some algebra,

$$mkw \leqslant (km - k + s)f(0) + kf(N) + (km + s - 1)\epsilon - sw$$

or, since $km + s = N$,

$$w \leqslant \frac{N-k}{N} f(0) + \frac{k}{N} f(N) + \frac{N-1}{N}\epsilon$$

By symmetry we can write also

$$w \geqslant \frac{N-k}{N} f(0) + \frac{k}{N} - \frac{N-1}{\epsilon}$$

and together these establish

$$\left| \frac{N-k}{N} f(0) + \frac{k}{N} f(N) - w \right| \leqslant \frac{N-1}{N}\epsilon \quad \square$$

Corollary 1. The theorem holds also if f is a complex function.

Proof. To show that

$$\left| f(k) - \frac{N-k}{N} f(0) - \frac{k}{N} f(N) \right| \leqslant \epsilon$$

multiply f by the appropriate phase to render $f(k)$ real and apply the theorem to the real part of f. $\square$

Remark. By the Hahn–Banach theorem the range of f could be any Banach space.

Corollary 2. If $\rho: \mathbf{Q} \times \mathbf{Q} \to \mathbf{C}$ is continuous and $\mu: \mathbf{Q} \to \mathbf{C}$ is as in (2.6), then μ is completably continuous.

Proof. Let $\xi \in \mathbf{R}$. By Lemma 1 there are $a, b \in \mathbf{Q}$, $a < \xi < b$ such that μ is 1-PA on $[a, b]$. Since every point of $[a, b] \subset \mathbf{Q}$ lies on some finite grid with endpoints a, b, it follows from Corollary 1 that μ can vary by no more than ± 1 on $[a, b]$, which contradicts Lemma 2 unless μ is completable at ξ. $\square$

Suppose finally that ρ has been derived from a multiplier σ. Completing μ to a continuous function $\bar{\mu}: \mathbf{R} \to \mathbf{C}$ one easily traces back through the relations (2.7), (2.8), (1.4) [with $G = \mathbf{R}(+)$] to show that $\lambda(x) \equiv e^{\bar{\mu}(x)}$ is a continuous remultiplier for σ.

3. SUGGESTIONS FOR THE INFINITE-DIMENSIONAL CASE

I have tried to formulate the above so that $\mathbf{R}$ can be replaced with an infinite-dimensional Abelian group, for example, Hilbert space H. Assuming that σ is *symmetric* as well as continuous, there will exist μ which, as before, may be discontinuous. Now, however, there is no dense analog of $\mathbf{Q}$ on which μ must be completably continuous. However, μ will still be locally ϵ-PA (Lemma 1) and one can hope to generalize the theorem of Section 2 as follows:

"If f is ϵ-PA on an appropriate subset S of H, then it is within ϵ of some PA function f_0 on S. (Thus "PA" generalizes "$= Ax + B$".)"

If the appropriate sets S include open sets (or, analogous to Q, sets that are dense in some open set), this generalization would furnish, for μ in some open set U (or S dense in U), a PA function μ_0 which would extend to all of H (or a dense subset thereof) by the PA property. Since μ_0 would be PA, $\hat{\mu} := \mu - \mu_0 + \mu_0(0)$ would be equivalent to μ but bounded in the open set U. By Lemma 2, $\hat{\mu}$ would be continuous (or completably continuous) as desired.

APPENDIX: FORMULA FOR λ ON Q

We remark parenthetically that the argumentation of Section 2 would have worked, not only for $\mathbf{Q}$, but for any dense subgroup of $\mathbf{R}$ which is the union of an increasing sequence of cyclic subgroups. For example, the group $\{m2^{-m}: m, n \in \mathbf{Z}\}$ is such a group, which is in some ways simpler than $\mathbf{Q}$. Thus the formula derived below would look somewhat simpler for it.

To construct a fairly explicit formula for λ on $\mathbf{Q}$ consider σ to be the multiplier of a representation R of $\mathbf{Q}$ as described at the beginning of Section 2. From (1.1) we can define functions $\sigma_n: \mathbf{Q}^n \to \mathbf{C}$ by

$$R(x_1)R(x_2)\cdots R(x_n) = \sigma(x_1, x_2, \ldots, x_n)R(x_1 + x_2 + \cdots + x_n) \quad (2.12)$$

(Thus $\sigma_2 \equiv \sigma$.) From (1.1), (1.2), and $\hat{R}(1/2)\hat{R}(1/2) = \hat{R}(1)$ follows [since $\lambda(1) = 1$]

$$\sigma(1/2, 1/2) = \lambda(1/2)^2 \qquad \Rightarrow \lambda(1/2) = \sigma(1/2, 1/2)^{1/2}$$

Similarly, $\hat{R}(1/6)^3 = \hat{R}(1/2)$ gives

$$\lambda(1/6) = \lambda(1/2)^{1/3}\, \sigma_3(1/6, 1/6, 1/6)^{1/3} = \sigma(1/2, 1/2)^{1/6}\sigma_3(1/6, 1/6, 1/6)^{1/3}$$

Continuing inductively, and defining for convenience

$$f(n, m) := \sigma_m(1/n, 1/n, \ldots, 1/n)$$

we get

$$\lambda(1/n!) = \prod_{k=2}^{n} f(k!, k)^{(k-1)!/n!}$$

where, of course, the arbitrary phases in $f(\ ,\ )^{1/n!}$ must be chosen consistently for all n.

Finally, $\hat{R}(1/n!)^m = \hat{R}(m/n!)$ shows that

$$\lambda(m/n!) = \lambda(1/n!)^m / f(n!, m)$$

In terms of ρ, μ, and $g := \log f$, we get [with $\mu(1) = 0$]

$$\mu(m/n!) = \frac{m}{n!} \sum_{k=2}^{n} (k-1)!\, g(k!, k) - g(n!, m)$$

in which all ambiguity has disappeared.

We have not given an explicit form for f in terms of σ, but this follows directly (if not uniquely) from (1.1) and (2.12). For example,

$$\sigma_3(x, x, x) = \sigma(x, x)\sigma(2x, x)$$

REFERENCES

Bargmann, V. (1954). *Annals of Mathematics*, **50**, 1.

Jauch, J. M. (1968). *Foundations of Quantum Mechanics*, Sections 12.5 and 13.4, Addison-Wesley, Reading, Mass.

McCarthy, P. J. (1977). "Lifting of projective representations of the Bondi–Metzner–Sachs Group," *Proceedings of the Royal Society*, **A358**, 141.

Weyl, H. (1950). *The Theory of Groups and Quantum Mechanics*, Chapter IV, Section D, Dover, New York.